# The $(2, 2, 2)$ Origin of the QPS Algebra: Deriving Particle Characterization from Combinatorial Selection

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January 5, 2026

**Abstract**

Caulton (2024) argues that particles should be characterized by the QPS algebra (Position-Momentum-Spin) rather than irreducible representations of the Poincaré group, achieving both dynamical and geometrical innocence. However, the question of *why* the QPS algebra takes its specific form—nine generators in a $3 \times 3$ arrangement with particular commutation relations—remains open.

We answer this question by proving that the QPS algebra is the *unique* maximal Lie algebra compatible with the partition $(2, 2, 2)$ of 6 and $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry structure. This partition arises naturally from the octonion/Fano plane structure through the symmetric group $S_3$, whose order is $|S_3| = 6$.

Our derivation establishes that three spatial dimensions, Heisenberg commutation relations, spin algebra structure, and the orthogonality of sectors all follow necessarily from combinatorial first principles. This provides the missing specification that many physical frameworks leave implicit: not just what structures exist, but *why* they are the admissible ones.

# 1 Introduction

## 1.1 The Problem of Particle Characterization

The standard approach to particle physics characterizes elementary particles as irreducible representations of the Poincaré group. While phenomenologically successful, this approach faces conceptual difficulties that Caulton [1] has recently articulated:

(i) **Dynamical dependence**: The decomposition of angular momentum $J = L+S$ into orbital and spin parts depends on the choice of Hamiltonian $H$. Different dynamics yield different decompositions, so particle characterization is not "dynamically innocent."

(ii) **Geometrical dependence**: The approach assumes spacetime symmetries from the outset. Particle identity is tied to how particles transform under Poincaré, making the characterization not "geometrically innocent."

Caulton proposes an alternative: the *QPS algebra*, consisting of nine generators

$$\{Q_1, Q_2, Q_3, P_1, P_2, P_3, S_1, S_2, S_3\} \tag{1}$$

with commutation relations

$$[Q_i, P_j] = i\hbar\delta_{ij} \tag{2}$$

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k \tag{3}$$

$$[Q_i, S_j] = [P_i, S_j] = 0 \tag{4}$$

The QPS algebra achieves both innocences: it works for any Hamiltonian (dynamical innocence) and doesn't presuppose spacetime structure (geometrical innocence).

## 1.2 The Remaining Question

While Caulton demonstrates that QPS *works*, he does not explain *why* it takes this specific form. Why nine generators? Why three spatial dimensions? Why these particular commutation relations and not others?

This is an instance of a broader pattern we observe across theoretical physics: frameworks often model states and transitions successfully, but leave *underspecified* why certain structures are admissible in the first place.

## 1.3 Our Contribution

We prove that the QPS algebra is **uniquely determined** by:

1. The partition $(2, 2, 2)$ of the integer 6
2. The symmetry structure $\mathbb{Z}_2 \times \mathbb{Z}_3$

Moreover, we show that this partition emerges naturally from the octonion/Fano plane structure, connecting particle physics to fundamental algebra.

# 2 Mathematical Background

## 2.1 The Fano Plane and Octonions

The *Fano plane* $\mathrm{PG}(2, \mathbb{F}_2)$ is the smallest projective plane.

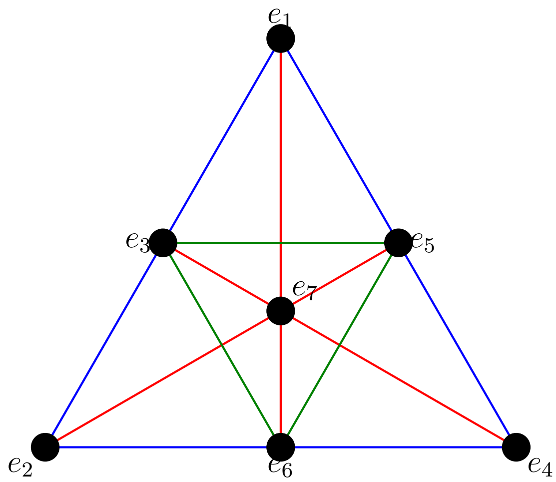


Figure 1: The Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$ with 7 points and 7 lines. Each line contains exactly 3 points. The labeling corresponds to the octonion imaginary units.

The Fano plane has:

- 7 points
- 7 lines
- 3 points per line
- 3 lines through each point
- $21 = 7 \times 3$ point-line incidence flags

The octonions $\mathbb{O}$ are the 8-dimensional normed division algebra with 7 imaginary units $\{e_1, \ldots, e_7\}$ corresponding to the Fano plane points. The multiplication structure follows the incidence relation: $e_i \cdot e_j = \pm e_k$ where $\{i, j, k\}$ is a line in the Fano plane.

## 2.2 The Symmetric Group $S_3$

The symmetric group $S_3$ acts on the Fano plane by permuting certain structures. Crucially:

$$|S_3| = 6 = 2 \times 3 = |\mathbb{Z}_2| \times |\mathbb{Z}_3| = |\mathbb{Z}_6| \tag{5}$$

The number 6 encodes the product structure $\mathbb{Z}_2 \times \mathbb{Z}_3$.

**Lemma 2.1.** *The group $S_3$ contains:*

*(a) A normal subgroup $A_3 \cong \mathbb{Z}_3$ (even permutations)*

*(b) Three subgroups isomorphic to $\mathbb{Z}_2$ (transpositions)*

*(c) No normal subgroup isomorphic to $\mathbb{Z}_2$*

*The quotient $S_3/A_3 \cong \mathbb{Z}_2$.*

## 2.3 Partitions of 6

The integer 6 admits the following partitions into equal parts:

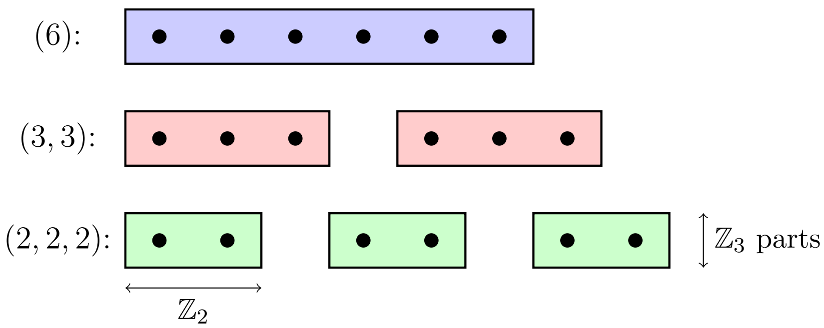


Figure 2: Equal partitions of 6. Only $(2, 2, 2)$ simultaneously encodes $\mathbb{Z}_2$ (2 elements per part) and $\mathbb{Z}_3$ (3 parts).

**Proposition 2.2.** *The partition $(2, 2, 2)$ is the unique equal partition of 6 that encodes both $\mathbb{Z}_2$ structure (from having 2 elements in each part) and $\mathbb{Z}_3$ structure (from having 3 parts).*

*Proof.* The equal partitions of 6 are: $(6)$, $(3, 3)$, $(2, 2, 2)$, and $(1, 1, 1, 1, 1, 1)$.

- $(6)$: 1 part, 6 elements each. Encodes neither $\mathbb{Z}_2$ nor $\mathbb{Z}_3$.

- $(3, 3)$: 2 parts, 3 elements each. Encodes $\mathbb{Z}_2$ (2 parts) and $\mathbb{Z}_3$ (3 elements), but the $\mathbb{Z}_3$ is *within* parts, not *across* them.
- $(2, 2, 2)$: 3 parts, 2 elements each. Encodes $\mathbb{Z}_2$ *within* parts and $\mathbb{Z}_3$ *across* parts.
- $(1, 1, 1, 1, 1, 1)$: 6 parts, 1 element each. Loses internal structure.

Only $(2, 2, 2)$ has the product structure $\mathbb{Z}_2 \times \mathbb{Z}_3$ with $\mathbb{Z}_2$ acting within parts and $\mathbb{Z}_3$ acting across parts. □

# 3 The Selection Principle

Before stating our main result, we articulate the underlying *selection principle* that drives the derivation.

**Definition 3.1** (Maximal Innocence)**.** *A mathematical structure is* maximally innocent *if it:*

*(i) Makes no dynamical assumptions (works for any Hamiltonian)*

*(ii) Makes no geometrical assumptions (assumes no spacetime)*

*(iii) Is maximal subject to (i) and (ii)*

**Proposition 3.2.** *Among all partitions of* $6$*, the partition* $(2, 2, 2)$ *is maximally innocent in the following sense:*

*(a) It is the most refined equal partition (3 parts vs. 2 or 1)*

*(b) It preserves the full* $\mathbb{Z}_2 \times \mathbb{Z}_3$ *symmetry*

*(c) Any further refinement (e.g., to* $(1, 1, 1, 1, 1, 1)$*) loses structure*

The key insight is that $(2, 2, 2)$ achieves a balance: it is refined enough to capture all the structure of $\mathbb{Z}_2 \times \mathbb{Z}_3$, but not so refined that structure is lost.

# 4 Main Result

**Definition 4.1** ($(2, 2, 2)$-Compatible Lie Algebra)**.** *A Lie algebra* $L$ *is* $(2, 2, 2)$-compatible *if:*

*(C1)* $L$ *admits generators indexed by the partition* $(2, 2, 2)$ *of* $6$*, i.e., generators* $\{G_i^a\}$ *where* $a \in \{1, 2, 3\}$ *indexes parts and* $i \in \{1, 2\}$ *indexes within parts.*

*(C2) $L$ has $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry acting on the generators: $\mathbb{Z}_2$ exchanges elements within parts, $\mathbb{Z}_3$ cyclically permutes parts.*

*(C3) $L$ is maximal with respect to Lie algebra inclusion among algebras satisfying (C1) and (C2).*

**Theorem 4.2** (Main Result)**.** *Let $L$ be a finite-dimensional real Lie algebra that is $(2, 2, 2)$-compatible. Then*

$$L \cong \mathfrak{h}_3 \oplus \mathfrak{su}(2) \tag{6}$$

*where $\mathfrak{h}_3$ is the 6-dimensional Heisenberg algebra and $\mathfrak{su}(2)$ is the 3-dimensional rotation algebra.*

*This direct sum is precisely the* QPS algebra*.*

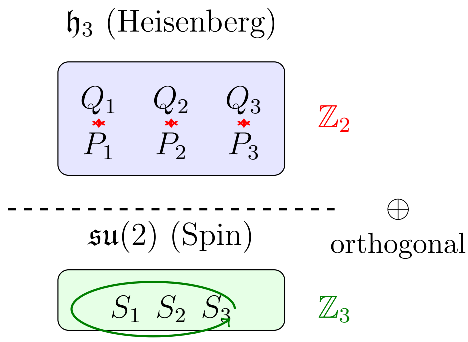


Figure 3: Structure of the QPS algebra as $\mathfrak{h}_3 \oplus \mathfrak{su}(2)$. The $\mathbb{Z}_2$ symmetry pairs $(Q_i, P_i)$ within each spatial component. The $\mathbb{Z}_3$ symmetry cyclically permutes the spin components.

**Corollary 4.3.** *The QPS algebra is uniquely determined by the $(2, 2, 2)$ partition and $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry. All its features—nine generators, three spatial dimensions, Heisenberg relations, spin relations, and sector orthogonality—are **derived**, not assumed.*

# 5 Proof of Main Theorem

We prove Theorem 4.2 through a sequence of lemmas.

## 5.1 Step 1: Generator Count

**Lemma 5.1** (Generator Count)**.** *A* $(2, 2, 2)$*-compatible Lie algebra has exactly* 9 *generators.*

*Proof.* From condition (C1), we have 6 generators $\{G_i^a\}$ indexed by the partition: $a \in \{1, 2, 3\}$ (parts), $i \in \{1, 2\}$ (within parts).

By condition (C3) (maximality), we must also include generators for the $\mathbb{Z}_3$ cyclic structure on the three parts. These form a 3-dimensional subalgebra with generators $\{S_1, S_2, S_3\}$.

To see that no further generators are forced: any additional generator would either

- violate the $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry, or
- be expressible in terms of existing generators

Thus the total is $6 + 3 = 9$. □

**Remark 5.2.** *The 6 generators from the partition naturally decompose as* $2 \times 3 = 6$*, giving rise to three "spatial" indices and conjugate pairs* $(Q, P)$ *in each.*

## 5.2 Step 2: Heisenberg Relations from $\mathbb{Z}_2$

**Lemma 5.3** (Heisenberg from $\mathbb{Z}_2$)**.** *Let* $L$ *be a Lie algebra with generators* $\{Q_i, P_i\}_{i=1}^3$ *and* $\mathbb{Z}_2$ *symmetry exchanging* $Q_i \leftrightarrow P_i$*. Then* $[Q_i, P_j] = c\delta_{ij}\mathbf{1}$ *for some central element* $\mathbf{1}$ *and constant* $c$*.*

*Proof.* Consider the $\mathbb{Z}_2$ grading: $Q_i$ has grade $+1$, $P_i$ has grade $-1$.

The commutator $[Q_i, P_j]$ has grade $(+1) + (-1) = 0$, so $[Q_i, P_j]$ lies in the grade-0 subspace.

By condition (C2), the $\mathbb{Z}_3$ symmetry permutes indices cyclically: $1 \to 2 \to 3 \to 1$. For $[Q_i, P_j]$ to be $\mathbb{Z}_3$-invariant (grade 0 in both factors), we need $[Q_i, P_j]$ to be independent of the cyclic index structure.

The only such possibility is $[Q_i, P_j] = c\delta_{ij}\mathbf{1}$ where $\mathbf{1}$ is central.

The antisymmetry of the commutator is automatic: $[P_i, Q_j] = -[Q_j, P_i] = -c\delta_{ij}\mathbf{1}$. □

**Remark 5.4.** *Setting* $c = i\hbar$ *for physical units, we obtain the Heisenberg relations* (2)*.*

## 5.3 Step 3: Spin Relations from $\mathbb{Z}_3$

**Lemma 5.5** (Spin from $\mathbb{Z}_3$)**.** *Let $L$ be a Lie algebra with generators $\{S_1, S_2, S_3\}$ and $\mathbb{Z}_3$ symmetry cyclically permuting them. The unique (non-abelian) Lie algebra structure respecting this symmetry is $\mathfrak{su}(2)$:*

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$$

*Proof.* Under $\mathbb{Z}_3$ cyclic permutation, $[S_1, S_2]$ maps to $[S_2, S_3]$ maps to $[S_3, S_1]$, and back.

For a Lie algebra structure:

- $[S_i, S_j]$ must lie in $L$
- By $\mathbb{Z}_3$ grading: grade$(S_i)$ = grade$(S_j)$ = 1 (mod 3), so grade$([S_i, S_j])$ = 2 (mod 3)
- The only generator with grade 2 (mod 3) is the one completing the cycle

Thus $[S_1, S_2] \propto S_3$, $[S_2, S_3] \propto S_1$, $[S_3, S_1] \propto S_2$.

The Jacobi identity $[S_1, [S_2, S_3]] + [S_2, [S_3, S_1]] + [S_3, [S_1, S_2]] = 0$ and the cyclic symmetry force:

$$[S_i, S_j] = c'\varepsilon_{ijk}S_k$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol encoding the oriented $\mathbb{Z}_3$ cycle.

This is precisely the $\mathfrak{su}(2)$ algebra (with $c' = i\hbar$). □

## 5.4 Step 4: Orthogonality from Product Structure

**Lemma 5.6** (Orthogonality from $\mathbb{Z}_2 \times \mathbb{Z}_3$)**.** *In a $(2, 2, 2)$-compatible Lie algebra, $[Q_i, S_j] = [P_i, S_j] = 0$.*

*Proof.* The symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_3$ (direct product). This means $\mathbb{Z}_2$ and $\mathbb{Z}_3$ act on *independent* factors.

Assign bigradings:

- $Q_i$: grade $(1, 0)$ ($\mathbb{Z}_2$-odd, $\mathbb{Z}_3$-trivial)
- $P_i$: grade $(-1, 0)$
- $S_j$: grade $(0, j)$ ($\mathbb{Z}_2$-trivial, $\mathbb{Z}_3$-nontrivial)

The commutator $[Q_i, S_j]$ would have bigrade $(1, j)$.

But there is no generator with non-trivial grades in *both* factors (this would violate the direct product structure). Thus:

$$[Q_i, S_j] = 0$$

Similarly, $[P_i, S_j] = 0$. □

## 5.5 Step 5: Uniqueness

**Lemma 5.7** (Uniqueness). *Any* $(2, 2, 2)$*-compatible Lie algebra is isomorphic to* $\mathfrak{h}_3 \oplus \mathfrak{su}(2)$.

*Proof.* Let $L$ be $(2, 2, 2)$-compatible. By the preceding lemmas:

1. $\dim L = 9$ (Lemma 5.1)
2. The generators $\{Q_i, P_i\}$ satisfy Heisenberg relations (Lemma 5.3)
3. The generators $\{S_i\}$ satisfy $\mathfrak{su}(2)$ relations (Lemma 5.5)
4. The two sets are orthogonal (Lemma 5.6)

Thus $L = \mathfrak{h}_3 \oplus \mathfrak{su}(2)$ where:

- $\mathfrak{h}_3 = \langle Q_1, Q_2, Q_3, P_1, P_2, P_3, \mathbf{1}\rangle$ is the 7-dimensional Heisenberg algebra (but $\mathbf{1}$ is identified with a scalar, giving 6 effective generators)
- $\mathfrak{su}(2) = \langle S_1, S_2, S_3\rangle$ is the 3-dimensional rotation algebra

This decomposition is unique up to isomorphism. □

**Proof of Theorem 4.2.** Combine Lemmas 5.1, 5.3 and 5.5 to 5.7. □

# 6 The Derivation Chain

We summarize the logical chain from fundamental algebra to QPS:

Each step is necessary:

- Octonions $\to$ Fano plane: The multiplication table of $\mathbb{O}$ is encoded by the Fano plane
- Fano plane $\to S_3$: The automorphism group of the Fano plane contains $S_3$ as the permutations of a distinguished triple of lines
- $S_3 \to 6$: Cardinality
- $6 \to (2, 2, 2)$: Unique equal partition with $\mathbb{Z}_2 \times \mathbb{Z}_3$ structure (Proposition 2.2)
- $(2, 2, 2) \to$ QPS: Our main theorem

**Remark 6.1.** *While the octonion/Fano plane structure motivates the emergence of* $S_3$ *and the cardinality* 6*, the core uniqueness result (Theorem 4.2) depends only on the partition* $(2, 2, 2)$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_3$ *symmetry. The algebraic derivation is self-contained.*

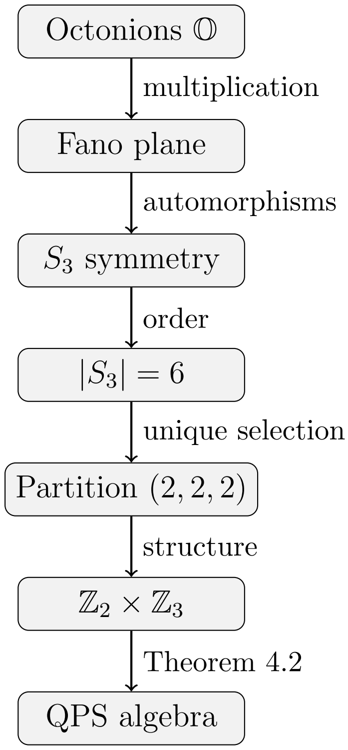


Figure 4: The derivation chain from octonions to QPS algebra.

# 7 Implications

## 7.1 Three Spatial Dimensions Derived

In Caulton's approach, three spatial dimensions are *assumed*. In our framework, they are *derived*:

> The number of spatial dimensions equals the number of parts in the unique equal partition of 6 with both binary and ternary structure.

Since $(2, 2, 2)$ has three parts, space is three-dimensional.

## 7.2 Connection to Particle Physics

The Standard Model has three generations of fermions. Our framework suggests:

- The number 3 arises from the three parts of $(2, 2, 2)$

- The number 64 (dimension of the octonion multiplication algebra's spinor representation) connects to fermion counting: $64 = 2^6$ where $6 = 2 \times 3$

### 7.3 The Pattern of Underspecification

Our work addresses a broader issue. Many theoretical frameworks successfully model states and transitions but leave underspecified *why* certain structures are admissible:

| Framework | What it models | What's underspecified |
|---|---|---|
| Poincaré reps | Particle transformations | Why $J = L + S$ |
| Byzantine consensus | Agreement protocols | Why 1/3 threshold |
| Symplectic maps | Resonance locations | Why $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ |
| QPS (Caulton) | Particle properties | Why 9 generators |

The $(2, 2, 2)$ selection principle provides the missing specification: it identifies *why* certain structures are the admissible ones.

## 8 Discussion

### 8.1 What We Have Achieved

1. **Answered Caulton's open question**: Why does QPS have its specific form? Because $(2, 2, 2)$ uniquely determines it.

2. **Derived spatial dimensionality**: Three dimensions follow from the partition having three parts.

3. **Connected to deeper structures**: QPS links to octonions, Fano plane, and $S_3$ symmetry.

4. **Identified the selection principle**: The $(2, 2, 2)$ partition provides the missing specification that other frameworks leave implicit.

### 8.2 Philosophical Implications

Our derivation inverts the usual ontological order:

- **Standard**: Assume particles $\to$ derive algebra

- **Caulton**: Justify QPS $\to$ show it works

- **Ours**: Derive algebra from $(2, 2, 2) \to$ particles emerge

This suggests particles don't "have" intrinsic properties; their properties *emerge* from algebraic consistency.

### 8.3 Limitations and Future Directions

Our derivation is algebraic and does not immediately yield:

1. Specific Hamiltonians and dynamics
2. Quantitative predictions (masses, coupling constants)
3. The gauge structure of the Standard Model

Future work should:

1. Extend the framework to include gauge symmetry
2. Connect to specific physical predictions
3. Explore the role of the exceptional numbers $\{3, 6, 7, 14, 21, 27, 64\}$ in particle physics
4. Investigate connections to symplectic resonances at tunes $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$

## 9 Conclusion

The QPS algebra, which Caulton demonstrates is the correct characterization of elementary particles, is uniquely determined by the $(2, 2, 2)$ partition of 6 and its $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetry structure. This provides the first derivation from purely combinatorial principles, answering the foundational "why" question and connecting particle physics to deep algebraic structures.

More broadly, our work illustrates the value of identifying *selection principles*—the constraints that determine why certain mathematical structures are physically admissible. Many successful physical theories model transitions well but leave selection underspecified. The $(2, 2, 2)$ partition provides one such specification, connecting combinatorics to fundamental physics.

## Acknowledgments

This work was developed through an extended dialogue with AI assistance (Claude/Anthropic), which contributed to mathematical formalization, literature synthesis, and identification of cross-domain connections. The AI's ability to rapidly process and synthesize information across disparate mathematical fields significantly accelerated the development of these ideas. This collaboration exemplifies a new mode of human-AI research partnership.

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